Entropy Computations via Analytic Depoissonization: Towards Analytic Information Theory

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Formulation of The Problem

Let $p_{n,k} = \Pr\{X_n = k\}$ be as sequence of distribution functions of discrete random variables X_n. For example:

- *Binomial distribution* defined for $k = 0, 1, \ldots, n$

$$p_{n,k} = \binom{n}{k} p^k (1-p)^{n-k}.$$

- *Negative binomial distribution* defined for for $n \geq 1$ and $k \geq n$ as

$$p_{n,k} = \binom{k}{n} p^{n+1} q^{k-n}.$$

Our main goal is to provide a (an analytic) tool that allows to derive a full asymptotic expansion for Rényi entropy

$$h_n(\omega) = \frac{1}{1-\omega} \ln \left(\sum_{k \geq 0} p_{n,k}^{\omega} \right),$$

We concentrate on Shannon entropy

$$h_n = \lim_{\omega \to 1} h_n(\omega),$$

and the binomial distribution.

First-Order Asymptotics

We assume:

(A1) Let $n\alpha$ and $n\sigma^2$ be the mean and the variance of the distribution $p_{n,k}$, where α and $\sigma^2 > 0$ are some constants. We postulate that for $k = n\alpha + x\sqrt{n}$ as $n \to \infty$

$$p_{n,k} = \frac{1}{\sqrt{2\pi n\sigma^2}} \exp\left(-\frac{k - n\alpha}{2n\sigma^2}\right)\left(1 + O\left(\frac{x^3}{\sqrt{n}}\right)\right) \ ,$$

where $x = o(n^{\frac{1}{6}})$, that is, $p_{n,k}$ converges with rate $O(x^3 n^{-1/2})$ to the normal distribution with mean $n\alpha$ and variance $n\sigma^2$.

(A2) For any $\varepsilon > 0$ and complex ω such that $\Re(\omega) > 0$ the following holds:

$$\sum_{|k-n\alpha|>n^{\frac{1}{2}+\varepsilon}} p_{n,k}^{\omega} = o(n^{-(\omega-1)/2})$$

as $n \to \infty$.

First-Order Asymptotics: Derivation

For any $0 < \varepsilon < \frac{1}{6}$

$$\exp\left((1-\omega)h_n(\omega)\right) = \sum_{|k-n\alpha|\le n^{\frac{1}{2}+\varepsilon}} p_{n,k}^{\omega} + o(n^{-(\omega-1)/2})$$

$$\left(1+O\left(n^{-\frac{1}{2}+\varepsilon}\right)\right)\frac{1}{(2\pi n\sigma^2)^{\omega/2}}\int_{-n^{\frac{1}{2}+\varepsilon}}^{n^{\frac{1}{2}+\varepsilon}} e^{-\frac{y^2\omega}{2n\sigma^2}}dy + o(n^{-(\omega-1)/2})$$

$$= \left(1+O\left(n^{-\frac{1}{2}+\varepsilon}\right)\right)\frac{1}{(2\pi n\sigma^2)^{\omega/2}}\int_{-\infty}^{\infty} e^{-\frac{y^2\omega}{2n\sigma^2}}dy + o(n^{-(\omega-1)/2})$$

$$= \frac{1}{\sqrt{\omega}(2\pi n\sigma^2)^{\frac{\omega-1}{2}}}\left(1+O\left(n^{-\frac{1}{2}+\varepsilon}\right)\right) + o(n^{-(\omega-1)/2})$$

where the first line follows directly from (A2), second line from (A1), and the next line is a consequence of the following well known approximation

$$\int_{\theta}^{\infty} x^k e^{-tx^2}dx = O\left(e^{-\frac{1}{2}t\theta^2}\right)$$

for any k, where θ is a positive number.

First-Order Asymptotics: Main Results

To recover asymptotically Shannon entropy h_n we apply Cauchy's formula

$$h_n = \frac{1}{2\pi i} \oint \frac{h_n(\omega)}{1-\omega} d\omega$$

to the above expression, where the integration is on any loop encircling $\omega = 1$.

Theorem 1. *Under assumptions (A1) and (A2) the Rényi entropy of order ω attains the following first order asymptotic*

$$h_n(\omega) = \frac{\ln\left(\omega^{-\frac{1}{2}}(2\pi n\sigma^2)^{-\frac{\omega-1}{2}}\right)}{1-\omega} + o(1)$$

In particular, Shannon entropy becomes (as $\omega \to 1$)

$$h_n = \frac{1}{2}\ln n + \frac{1}{2}\left(1 + \ln(2\pi\sigma^2)\right) + o(1)$$

as $n \to \infty$.

Full Asymptotic Expansion

Consider

$$h_n = -\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \ln\left(\binom{n}{k} p^k q^{n-k}\right)$$

for large n, that is, Shannon entropy for the *binomial*(n,p) distribution.

Theorem 2. *Shannon entropy, h_n, of the binomial distribution possesses the following asymptotic expansion*

$$\begin{aligned} h_n &\asymp \frac{1}{2}\ln n + \frac{1}{2} + \frac{1}{2}\ln(2\pi pq) \\ &+ \sum_{m=1}^{\infty} \frac{1}{n^{2m-1}} \left(\frac{B_{2m}(p^{1-2m} + q^{1-2m} - 1)}{2m(2m-1)} + o_m\right) \\ &+ \sum_{m=1}^{\infty} \frac{e_m}{n^{2m}} \end{aligned}$$

where B_k are the Bernoulli numbers, $q = 1 - p$, while e_m and o_m are explicitly computable constants.

Main Ingredients of the Proof

1. STARTING ....
Re-write h_n as

$$\begin{aligned} h_n &= -\ln(n!) - n(p\ln p + q\ln q) \\ &+ \sum_{k=0}^{n} \ln(k!)\binom{n}{k}p^k q^{n-k} + \sum_{k=0}^{n} \ln((n-k)!)\binom{n}{k}p^k q^{n-k}. \end{aligned}$$

2. POISSONIZING ...
Let

$$g_n := \sum_{k=0}^{n} \ln(k!)\binom{n}{k}p^k q^{n-k} + \sum_{k=0}^{n} \ln((n-k)!)\binom{n}{k}p^k q^{n-k} \ ,$$

and define the **Poisson transform** of g_n as follows

$$\widetilde{G}(z) = \sum_{n=0}^{\infty} g_n \frac{z^n}{n!} e^{-z}$$

for z complex. We also write $\mathcal{P}(g_n; z) := \widetilde{G}(z)$ for Poisson transform.

In our case

$$\widetilde{G}(z) = \widetilde{F}(zp) + \widetilde{F}(zq)$$

where

$$\widetilde{F}(z) = \sum_{n=0}^{\infty} \ln(n!) \frac{z^n}{n!} e^{-z} \ .$$

Problem: We do NOT have an explicit formula on $\widetilde{F}(z)$ since there is no explicit formula for $f_n = \log(n!)$. But we know that

$$f_n = f(n) = \log \Gamma(n+1).$$

and we know asymptotic expansion of $\Gamma(z)$ for large z.

Solution: Analytic Poissonization. We develop a technique to obtain an asymptotic expansion of $\widetilde{F}(z) = \mathcal{P}(f_n; z)$ from the asymptotic expansion of the analytic continuation $f(z)$ of $f(n)$.

3. DEPOISSONIZING ...

Thus, we reduce the problem to the following computation

$$h_n = -\ln(n!) - n(p\ln p + q\ln q) + \mathcal{P}^{-1}\left(\widetilde{F}(zp) + \widetilde{F}(zq); n\right) .$$

where $\mathcal{P}^{-1}(\widetilde{G}(z); n) := g_n$ denotes (analytic) depoissonization.

Problem. How, knowing asymptotic expansion of $\widetilde{G}(z)$, one can recover asymptotically g_n. Examples suggest that

$$g_n \sim \widetilde{G}(n).$$

(Indeed, $\widetilde{G}(z) = \mathbf{E}[g_{N(z)}]$ where $N(z)$ is Poisson with parameter z. Since $N(z)$ is strongly concentrated around the mean $z = n$, hence one expect the above).

NOT TRUE ... in general. Take

$$g_n = (-1)^n.$$

Indeed, $\widetilde{G}(z) = e^{-2z}$ and $(-1)^n \not\sim \widetilde{G}(n)$.

Finally ...

In P. Jacquet & W. Szpankowski, "Analytical Depoissonization and Its Applications, *Theoretical Computer Science*, 201, No. 1-2, 1–62, 1998
we have developed a theory of analytic depoissonization (a tauberian kind theorem) that allows us to conclude that:

$$\begin{aligned} h_n &= -f(n) + f(np) + f(nq) - n(p\ln p + q\ln q) \\ &+ \sum_{i=1}^{\infty}\sum_{k=2i}^{\infty} n^i \left(c_{i,k}(p)p^i f^{\langle k\rangle}(np) + c_{i,k}(q)q^i f^{\langle k\rangle}(nq)\right). \end{aligned}$$

where $c_{i,k}(\gamma)$ are Taylor's coefficients of the following function

$$\exp\left(x\ln\left(1+\lambda(e^y-1)\right) - \lambda xy\right) = \sum_{i=1}^{\infty}\sum_{k=2i}^{\infty} c_{i,k}(\lambda)x^i y^k,$$

with λ being either equal to p or to q. The final result follows from this and (recall: $f(n) = \log n!$)

$$f^{\langle k\rangle}(z) = \frac{\mathrm{d}^k}{\mathrm{d}z^k}\Gamma(z+1) = \Psi^{(k-1)}(z+1) = \Psi^{(k-1)}(z) + \frac{(-1)^{k-1}k!}{z^k}.$$

Analytic Depoissonization

Let us consider a general situation. Assume a sequence, say g_n, is given, and its Poisson transform is $\widetilde{G}(z)$. We want to know simple conditions under which $g_n \sim \widetilde{G}(n)$ holds for $n \to \infty$.

In Jacquet&Szpankowski TCS 1998 we proved, among others, the following result:

Theorem 3. [Jacquet and Szpankowski 1998] *Consider a linear cone* $\mathcal{S}_\theta = \{z : \ |\arg(z)| \le \theta, \ \ \theta < \pi/2\}$*. Let the following two conditions hold for some numbers* A*,* B*,* $R > 0$ *and* $\alpha < 1$ *and* β*:*
(I) *For* $z \in \mathcal{S}_\theta$

$$|z| > R \quad \Rightarrow \quad |\widetilde{G}(z)| \le B|z|^\beta \Psi(|z|) \ ,$$

where $\Psi(x)$ *is a slowly varying function , that is, such that for fixed* t $\lim_{x\to\infty} \frac{\Psi(tx)}{\Psi(x)} = 1$ *(e.g.,* $\Psi(x) = \ln^d x$ *for some* $d > 0$*);*

(O) *For* z *outside the cone* $\mathcal{S}_\theta$*, that is, for* $z \notin \mathcal{S}_\theta$

$$|z| > R \quad \Rightarrow \quad |\widetilde{G}(z)e^z| \le A\exp(\alpha|z|)$$

Then, for every nonnegative integer m

$$\begin{aligned} g_n &= \sum_{i=0}^{m}\sum_{j=0}^{i+m} b_{ij} n^i \widetilde{G}^{\langle j\rangle}(n) + O(n^{\beta-m-1}\Psi(n)) \\ &= \widetilde{G}(n) + \sum_{k=1}^{m}\sum_{i=1}^{k} b_{i,k+i} n^i \widetilde{G}^{\langle k+i\rangle}(n) + O(n^{\beta-m-1}\Psi(n)) \end{aligned}$$

where $\widetilde{G}^{\langle j\rangle}(n)$ *is the* j*th derivative of* $\widetilde{G}(z)$ *at* $z = n$, *and* b_{ij} *are the coefficients of* $\exp(x\ln(1+y) - xy)$ *at* $x^i y^j$, *that is:*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{ij} x^i y^j = \exp(x\ln(1+y) - xy) \; .$$

such that $b_{ij} = 0$ *for* $j < 2i$. *In fact,* $b_{ij} = s_2(j,i)/j!$ *where* $s_2(n,k)$ *are the associated Stirling numbers of the first kind.*

Formal Derivation of the Coefficients b_{ij}

Observe that Taylor's expansion of $\widetilde{G}(z)$ around $z = n$ becomes

$$\widetilde{G}(z) = \sum_{k=0}^{\infty} \widetilde{G}^{\langle k \rangle}(n) \frac{(z-n)^k}{k!} .$$

We use formal power series approach to obtain b_{ij}. Since we want to identify the coefficient at n^i and $\widetilde{G}^{\langle k \rangle}(n)$ we formally define $x^i = n^i$ and $y^k := \widetilde{G}^{\langle k \rangle}(n)$, rearrange the series, and extract the coefficient at $x^i y^k$.

$$\widetilde{G}(z) = \sum_{k \geq 0} \frac{(z-n)^k}{k!} y^k = \exp(y(z-n)) \quad \text{where} \quad y^k = \widetilde{G}^{\langle k \rangle}(n) .$$

Thus,

$$\begin{aligned} g_n &= n![z^n](e^z \widetilde{G}(z)) = n![z^n](e^{-yn} e^{(1+y)z}) = (1+y)^n e^{-ny} \\ &= \exp(n \ln(1+y) - ny) \qquad \text{where} \qquad y^k = \widetilde{G}^{\langle k \rangle}(n) \end{aligned}$$

Analytic Poissonization

Let us now assume that g_n can be analytically continued to $g(z)$ and $g(z)$ has an asymptotic expansion for $z \to \infty$ in a cone. How can we get asymptotic expansion of $\widetilde{G}(z) = \mathcal{P}(g_n; z)$?

Theorem 4. *Let $g(z) = O(z^\beta)$ in a linear cone, where β is a constant. Then, for every nonnegative integer m and complex $w = \lambda z$ for some constant λ (more generally: $w = \Theta(z)$), as $z \to \infty$*

$$\begin{aligned}\widetilde{G}(z) &= \sum_{i=0}^{m}\sum_{j=0}^{i+m} a_{ij}(\lambda) z^i g^{\langle j\rangle}(w) + O(z^{\beta-m-1}) \\ &= g(w) + \sum_{k=1}^{m}\sum_{i=1}^{k} a_{i,k+i}(\lambda) z^i g^{\langle k+i\rangle}(w) + O(z^{\beta-m-1})\end{aligned}$$

where $a_{ij}(\lambda)$ are coefficients of $\exp(x(e^y - 1) - wy)$ at $x^i y^j$. More precisely: for $\lambda = 1$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} x^i y^j = \exp(x(e^y - 1) - xy) \ .$$

and $a_{ij} = 0$ for $j < 2i$. In fact, $a_{ij} = S_2(j,i)/j!$ where $S_2(n,k)$ are the 2-associated Stirling numbers of the second kind.

Proof and Remark

Remark. The above can be formally re-written as

$$\widetilde{G}(z) = \exp\left(z(e^y - 1) - wy\right) \quad \text{where} \quad y^j = g^{\langle j \rangle}(w) \; .$$

In addition,

$$\widetilde{G}^{\langle k \rangle}(z) = (e^y - 1)^k \exp\left(z(e^y - 1) - wy\right) \quad \text{where} \quad y^j = g^{\langle j \rangle}(w) \, .$$

for $w = \lambda z$.

Proof. We need to identify the coefficients at $z^i g^{\langle j \rangle}(w)$ and we shall use formal power series.

$$\begin{aligned}
\widetilde{G}(z) &= \sum_{n=0}^{\infty} g(n) \frac{z^n}{n!} e^{-z} = \sum_{n=0}^{\infty} \frac{z^n}{n!} e^{-z} \sum_{j=0}^{\infty} g^{\langle j \rangle}(w) \frac{(n-w)^j}{j!} \\
&= \sum_{n=0}^{\infty} \frac{z^n}{n!} e^{-z} \sum_{j=0}^{\infty} y^j \frac{(n-w)^j}{j!} \quad \text{where} \quad y^j = g^{\langle j \rangle}(w) \\
&= \sum_{n=0}^{\infty} \frac{z^n}{n!} e^{-z} e^{y(n-w)} = \exp\left(z(e^y - 1) - wy\right); y^j = g^{\langle j \rangle}(w) \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} z^i g^{\langle j \rangle}(w)
\end{aligned}$$

The error term is easy to prove to be $O(z^{\beta - m - 1})$.

Examples

For $m = 6$ we have

$$\begin{aligned} g_n \quad = \quad & \widetilde{G}(n) - \frac{1}{2}n\widetilde{G}^{\langle 2\rangle}(n) + \frac{1}{3}n\widetilde{G}^{\langle 3\rangle}(n) + \frac{1}{8}n^2\widetilde{G}^{\langle 4\rangle}(n) - \frac{1}{4}n\widetilde{G}^{\langle 4\rangle}(n) - \frac{1}{6}n^2\widetilde{G}^{\langle 5\rangle}(n) - \\ & -\frac{1}{48}n^3\widetilde{G}^{\langle 6\rangle}(n) + \frac{1}{5}n\widetilde{G}^{\langle 5\rangle}(n) + \frac{13}{72}n^2\widetilde{G}^{\langle 6\rangle}(n) + \frac{1}{24}n^3\widetilde{G}^{\langle 7\rangle}(n) + \frac{1}{384}n^4\widetilde{G}^{\langle 8\rangle}(n) - \\ & -\frac{1}{6}n\widetilde{G}^{\langle 6\rangle}(n) - \frac{11}{60}n^2\widetilde{G}^{\langle 7\rangle}(n) - \frac{17}{288}n^3\widetilde{G}^{\langle 8\rangle}(n) - \frac{1}{144}n^4\widetilde{G}^{\langle 9\rangle}(n) - \frac{1}{3840}n^5\widetilde{G}^{\langle 10\rangle}(n) \\ & +O(n^{\beta-6}) \ , \end{aligned}$$

and

$$\begin{aligned} \widetilde{G}(z) \quad = \quad & g(z) + \frac{1}{2}\, z\, g^{\langle 2\rangle}(z) + \frac{1}{6}\, z\, g^{\langle 3\rangle}(z) + \left(\frac{1}{24}\, z + \frac{1}{8}\, z^2\right)\, g^{\langle 4\rangle}(z) + \left(\frac{1}{120}\, z + \frac{1}{12}\, z^2\right)\, g^{\langle 5\rangle}(z) \\ + \quad & \left(\frac{1}{720}\, z + \frac{5}{144}\, z^2 + \frac{1}{48}\, z^3\right)\, g^{\langle 6\rangle}(z) + \left(\frac{1}{5040}\, z + \frac{1}{90}\, z^2 + \frac{1}{48}\, z^3\right)\, g^{\langle 7\rangle}(z) \\ + \quad & \left(\frac{1}{40320}\, z + \frac{17}{5760}\, z^2 + \frac{7}{576}\, z^3 + \frac{1}{384}\, z^4\right)\, g^{\langle 8\rangle}(z) \\ + \quad & \left(\frac{1}{362880}\, z + \frac{41}{60480}\, z^2 + \frac{137}{25920}\, z^3 + \frac{1}{288}\, z^4\right)\, g^{\langle 9\rangle}(z) + \mathrm{O}(\, z^{\beta-6}\,) \end{aligned}$$

Poly-Log Growth

In many applications $g(z)$ has poly-log growth. In some situation, verifying condition (O) (outside the cone) might be troublesome. The theorem below makes it easy!

Lemma 1. *Let $g(z)$ be an analytic continuation of a sequence g_n whose Poisson transform is $\widetilde{G}(z)$, and such that $g(z) = O(z^\beta)$ in a linear cone. Then, for some θ_0 and for all linear cones $\mathcal{S}_\theta$ ($\theta < \theta_0$), there exists $\alpha < 1$ and $A > 0$ such that*

$$z \notin \mathcal{S}_\theta \qquad \Rightarrow \qquad |\widetilde{G}(z)e^z| \le Ae^{\alpha|z|} .$$

Proof. We define Laplace transform of $g(z)$ as

$$\begin{aligned} g^*(s) &= \int_0^\infty g(x)e^{-sx}dx \\ g(x) &= \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} g^*(s)e^{sx}ds \end{aligned}$$

for $\Re(s) > 0$.

Observe that $G(z) = \widetilde{G}(z)e^z$ becomes

$$
\begin{aligned}
G(z) &= = \sum_{n=0}^{\infty} g(n)\frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} g^*(s)\exp(ns)\frac{z^n}{n!}ds \\
&= \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} g^*(s)\sum_{n=0}^{\infty}\exp(ns)\frac{z^n}{n!}ds \\
&= \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} g^*(s)\exp\left(ze^s\right)ds
\end{aligned}
$$

for $\varepsilon > 0$.

We want to prove that when $g(z) = O(z^\beta)$, then $|G(z)| \leq Ae^{\alpha|z|}$ for $\alpha < 1$.

Extension of Laplace to Complex Functions

We can prove that as long as $g(z)$ (a function of complex z) has a polynomial growth, then the Laplace $g^*(s)$ exists in a bigger cone $\mathcal{S}_{\theta+\pi/2}$ for all $\theta < \theta_0$, and then the inverse Laplace becomes

$$g(z) = \frac{1}{2\pi i} \int_{\mathcal{L}_\varepsilon} g^*(s) e^{sz} ds$$

where $\mathcal{L}_\varepsilon$ is a piece-linear curve that parallels the boundary of the cone $\mathcal{S}_{\theta+\pi/2}$ at distance ε.

Final Touches ....

We can now upper bound $\widetilde{G}(z)e^z$ as follows

$$
\begin{aligned}
|\widetilde{G}(z)e^z| &\leq \frac{1}{2\pi}\int_{\mathcal{L}_\varepsilon} |g^*(s)| \exp(\Re(e^s z)) ds \\
&\leq \frac{1}{2\pi}\int_{\mathcal{L}_\varepsilon} |g^*(s)| \exp\left(\Re(e^s e^{i\theta})|z|\right) ds
\end{aligned}
$$

since for $z \notin \mathcal{S}_\theta$ we have $\cos(\arg z) \leq \cos\theta$ for $|\theta| \leq \pi/2$.

To complete the proof, we shall show that

$$\Re(e^s e^{i\theta}) < \alpha \quad \text{for} \quad s \in \mathcal{L}_\varepsilon$$

for some $\alpha < 1$.

Provided it is true, we obtain immediately

$$|\widetilde{G}(z)e^z| \leq Ae^{\alpha|z|}$$

for $\alpha < 1$ where

$$A = \frac{1}{2\pi}\int_{\mathcal{L}_\varepsilon} |g^*(s)| ds.$$

Image of $\mathcal{L}_0$

Observe that

$$\widetilde{G}(z)e^z = \frac{1}{2\pi i}\int_{\mathcal{L}_\varepsilon} g^*(s)\exp[e^s e^{i\arg(z)}|z|]ds \ .$$

Let $\mathcal{I}_\varepsilon$ and $\mathcal{I}_0$ be images of $\mathcal{L}_\varepsilon$ and $\mathcal{L}_0$ over the function e^s.

For $\theta = \pi/6$ and $\varepsilon = 0.1$ it is shown below:

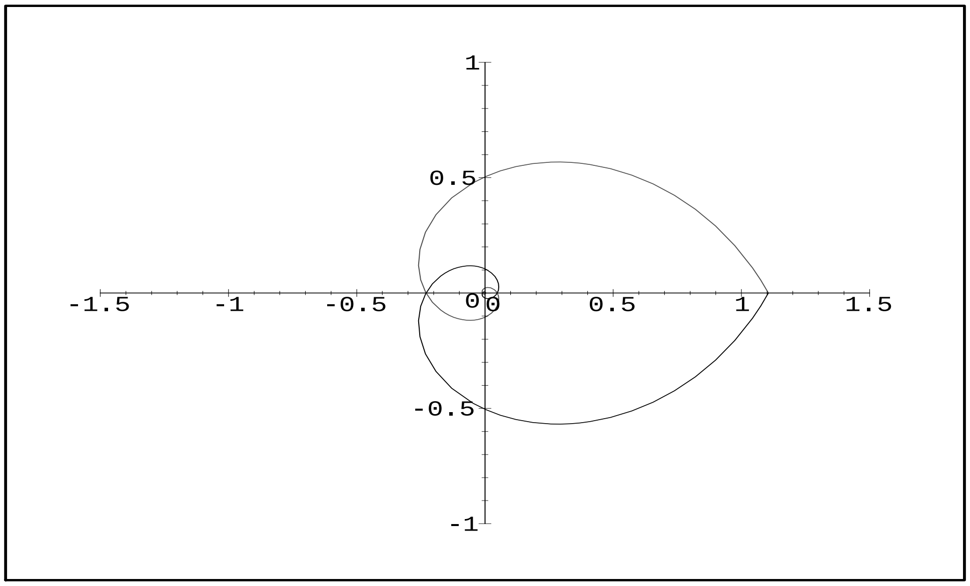

Figure 1: Image of $\mathcal{L}_\varepsilon$ with $\varepsilon = 0.1$ by e^s.

End of the Proof ...

Let us now fix an arbitrary $0 < \theta < \theta_0$, and consider for a moment only the image $\mathcal{I}_0$ of $\mathcal{L}_0$. Observe that if one rotates the image $\mathcal{I}_0$ by a non zero argument, then the new image has the real part strictly less than 1. That is, the image of $\mathcal{L}_\varepsilon$ over $e^{s+i\theta}$ gives us

$$\Re(e^s e^{i\theta}) < \alpha < 1$$

which completes the proof.

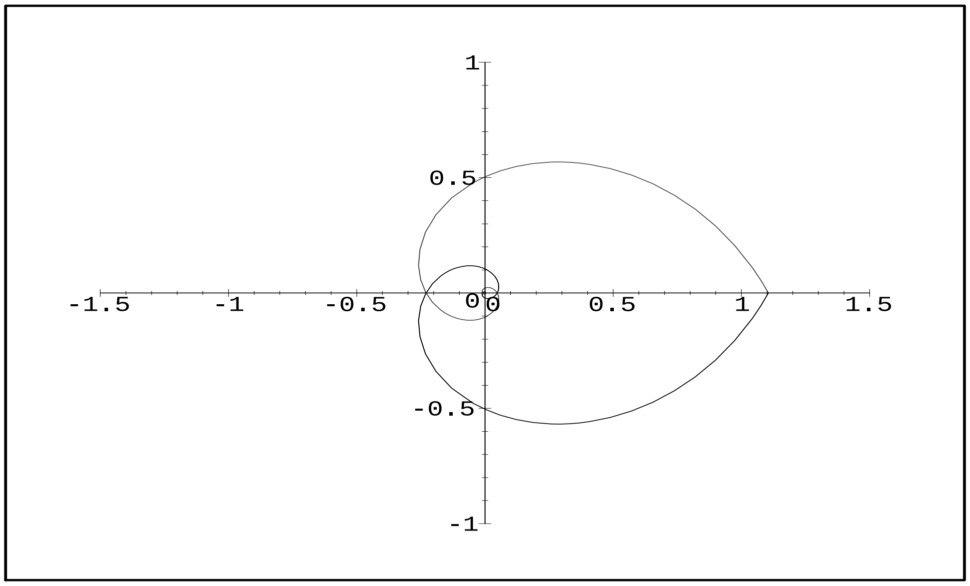

Figure 2: Image of $\mathcal{L}_\varepsilon$ with $\varepsilon = 0.1$ by e^s.

Binomial Sums

We should observe that we develop a method of obtaining asymptotic expansion of the so called binomial sum

$$S_n = \sum_{k=0}^{n} \binom{n}{k} p^k q^k f_k$$

where f_k is a sequence that can be analytically continued to a poly-log function $f(z)$. Recently, Flajolet proposed a technique based on the **singualrity analysis** to deal with binomial sums:

P. Flajolet, "Singularity Analysis and Asymptotics of Bernoulli Sums", *Theoretical Computer Science*, to appear.

Similar results to ours were obtained also in:

C. Knessl, Integral Representations and Asymptotic Expansions for Shannon and Renyi Entropies, *Appl. Math. Lett.*, 11, 69–74, 1998.

$$S_n^{\omega} = \sum_{k=0}^{n} \binom{n}{k}^{\omega} p^{k\omega} q^{k\omega} f_k \quad \omega > 0$$

Bonus I: Krichevskiy's Sum

Recently, Krichevskiy analyzed in IT98 the redundancy of the "add-β" rule for predicting the $(n+1)$st symbol. He comes up with the following function:

$$F(n,\beta,p) = \beta + np\ln(np) - np\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}\ln(k+\beta),$$

where $0 < p < 1$ and β is a constant. We can give a full asymptotic expansion.

Let

$$g_n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \ln(k+\beta),$$

then

$$\begin{aligned} g_n &= \sum_{i \ge 0} \sum_{j \ge 2i} c_{ij}(p) n^i f^{\langle j \rangle}(np), \qquad f(n) = \ln(n+\beta) \\ &= f(np) + \sum_{i \ge 1} \sum_{j \ge 2i} c_{ij}(p) n^i f^{\langle j \rangle}(np) \\ &= \ln(np+\beta) - \frac{q}{2np} + \sum_{k \ge 2} \frac{a_k(p,\beta)}{n^k}, \end{aligned}$$

and

$$F(n,\beta,p) = \frac{q}{2} - \sum_{k \ge 1} \frac{1}{n^k} \left(p a_{k+1}(p,\beta) + \frac{(-1)^k \beta^{k+1}}{(k+1)p^k} \right)$$

where $q = 1 - p$.

Bonus II: Redundancy of Fano-Shannon Code

When studying the average redundancy of Fano-Shannon code, one needs to analyze the following sum

$$r_n = \sum_{k=1}^{n} p_{n,k} \{\log(p^k (1-p)^{(n-k)})\}$$

where

$$p_{n,k} = \binom{n}{k} p^k (1-p)^{(n-k)}$$

is the binomial distribution and where

$$\{\log n\} = \log n - \lfloor \log n \rfloor$$

is the fractional part. We can give precise asymptotics of the redundancy r_n

Set $\alpha = \log(p/(1-p))$ and $\beta = \log(1-p)$. Then $\log(p^k(1-p)^{(1-k)}) = \alpha k + \beta n$.

Using the theory *uniform distribution of sequences*, M. Drmota suggested the following solution. If α is irrational, then

$$\lim_{n\to\infty} \sum_{k=1}^{n} p_{n,k} f(\alpha k + \beta n) = \int_0^1 f(t)dt$$

for all Riemann integrable real valued functions f with period 1, i.e $f(x+1) = f(x)$. Thus

$$r_n = \int_0^1 \{t\}dt + o(1) = \frac{1}{2} + o(1).$$

If $\alpha = a/b$ is rational then it is easy to see that

$$r_n = \frac{1}{b}\sum_{l=0}^{b-1}\{l/b+\beta n\}+o(1) = \frac{1}{2}+\frac{1}{b}B_1(\{bn\log(1-p)\})+o(1),$$

where $B_1(x) = x - 1/2$ is the first Bernoulli polynomial. So, the average redundancy is actually 1/2 not 1!

Analytic Information Theory?

Analytic number theory was born in 1896 when independently Hadamard and de la Vallée-Poussin proved the *prime number theory* through a sophisticated application of complex analysis methods.

Analytic average–case analysis of algorithms started more than thirty years ago by Knuth who used systematically generating functions to predict the performance of computer algorithms.

Flajolet, Odlyzko and Sedgewick launched **analytic combinatorics** by applying Mellin transform and the singularity analysis to combinatorial problems. In all of the above cases, theory of **analytic** functions was used to solve problems of diversified areas of science and engineering.

In view of this, it is natural to coin the term **analytic information theory** to describe problems of information theory that are solved by analytic methods, that is, those in which complex analysis plays a pivotal role.

P. Jacquet, *Eléments de Théorie Analytique de l'Information, Modélisation et Evaluation*, Habilitation, INRIA, France July 1998.